THE SOLUTION OF SEVERAL BOUNDARY PROBLEMS IN MAGNETOHYDRODYNAMICS

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PMM Vol.25, No.5, 1961, pp. 965-968

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(Received April 30, 1961)

The problems dealt with involve the flow of an electroconductive medium through a plane channel or duct in the presence of a magnetic field with various boundary conditions. The walls of the central part of the duct are electrodes and the rest of the walls are insulating. Solutions are obtained both for the case of a constant magnetic field and an arbitrary law of transverse variation in velocity, and also for a constant velocity of flow with arbitrary variation of magnetic field along the electrodes.

1. Fluid of constant electrical conductivity σ flows through a channel (Fig. 1) with plane walls $y = \pm\delta$, one section of which, *1*, is insulated; other sections, 2, symmetrical with respect to the channel axis, are electrodes $y = \pm\delta$, $a_\nu \leqslant x \leqslant b_\nu$, each νth pair of which is connected through external load $R_\nu (\nu = 1, \ldots n)$. Let the external magnetic field be $\mathbf{B} = (0, 0, -B(x))$, $B(x) \geqslant 0$ perpendicular to the plane of flow, and depending only on the coordinate axis x. The interaction of the fluid flow and the magnetic field involves an electric power load $N_\nu = J_\nu^2 R_\nu$, where J_ν is the total load current. If the magnetic Reynolds numbers are small (as is the case in many applications), the effect of the induced magnetic field on the flow may be neglected and the distribution of current density j and potential ϕ can be obtained from Ohm's Law and the equations of continuity

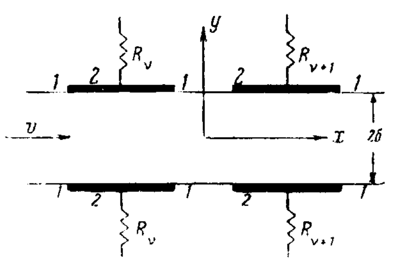

Fig. 1.

$$\mathbf{j} = \sigma\left(-\nabla\varphi + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right), \qquad \operatorname{div}\mathbf{j} = 0 \tag{1.1}$$

in which the magnetic field is assumed to be known.

To a first approximation, the stream velocity **v** may be considered a known quantity from hydrodynamic considerations. Equations (1.1), then, represent a closed system for determining the currents and potential. Assuming the velocity to have only a longitudinal component V, we can write down system (1.1) as follows:

$$\frac{1}{\sigma} j_x = -\frac{\partial \varphi}{\partial x}, \qquad \frac{1}{\sigma} j_y = -\frac{\partial \varphi}{\partial y} + \frac{1}{c} VB, \qquad \triangle \varphi = \frac{1}{c} \frac{\partial}{\partial y} (VB) \tag{1.2}$$

The function ϕ should satisfy the following boundary conditions:

$$\varphi(x, \pm \delta) = \pm \varphi_\nu \qquad \text{— on the electrodes} \tag{1.3}$$

$$\frac{\partial \varphi}{\partial y} = \frac{1}{c} VB \qquad \text{— on the insulators} \tag{1.4}$$

The constants ϕ_ν must be obtained by applying Ohm's Law to the external load

$$R_\nu \int_{a_\nu}^{b_\nu} j_y(x, \pm\delta) dx = 2\varphi_\nu \tag{1.5}$$

2. We will now deal with gas flow through a channel with a constant magnetic field. Assume the velocity to be some arbitrary even function of y. Then, introducing the function $u(x, y)$ through formula

$$u = \varphi - \frac{1}{c} B \int_0^y V dy$$

we obtain from the relations (1.2)

$$\frac{1}{\sigma} j_x = -\frac{\partial u}{\partial x}, \qquad \frac{1}{\sigma} j_y = -\frac{\partial u}{\partial y}, \qquad \triangle u = 0 \tag{2.1}$$

First of all we assume that the upper and the lower walls are insulating. The system (2.1) has a solution

$$u \equiv 0, \qquad \mathbf{j} \equiv 0, \qquad \varphi = \frac{1}{c} B \int_0^y V dy \tag{2.2}$$

In that case a separation of the electric charges takes place in the channel, with the result that a potential builds up between the upper and lower walls

$$\varphi(x, +\delta) - \varphi(x, -\delta) = \frac{1}{c} B \int_{-\delta}^{+\delta} V dy = \mathscr{E}$$

Now suppose that the wall sections $y = \pm\delta$, $a_\nu \leqslant x \leqslant b_\nu$ become electrodes.

Because the walls are insulating at infinity the previous solution may be used. Thus

$$\varphi(x, \pm\delta) = \pm{}^1/_2\,\mathscr{E}, \qquad u(x, \pm\delta) = 0 \quad \text{for } x \to \pm\infty$$

Introduce the analytic function $w(z) = u + iv$. At the electrodes the real part of this function is constant $u(x \pm \delta) = \mp(1/2\,\mathscr{E} - \phi_\nu) = \mp u_\nu$ ($u_\nu \geqslant 0$, for the potential drops with internal load). On the insulators the imaginary part $v(x, y)$ is constant. The jumps $v_\nu = v(b_\nu, \delta)$

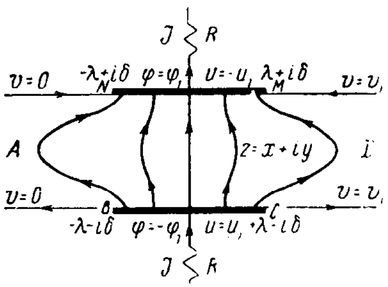

Fig. 2.

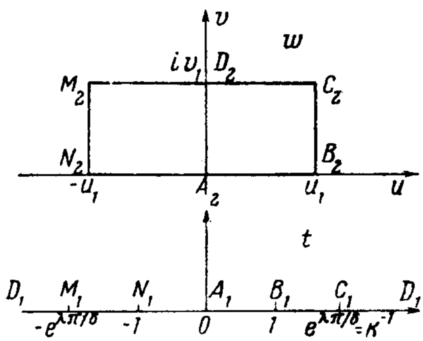

Fig. 3.

$- v(a_\nu, \delta)$ in the function $v(x, y)$ at the electrodes follow from Expression (1.5):

$$R_\nu \sigma v_\nu = \mathscr{E} - 2u_\nu \tag{2.3}$$

Thus the problem can be solved by conformal mapping of the regions $-\delta < \operatorname{Im} z < \delta$ onto the inside of a polygon in the w-plane whose sides are parallel to the coordinate axes, and the lengths of the sides are connected by relations (2.3); when $\nu \geqslant 2$, such a representation generally yields a definite relationship between the loads R_ν.

As an example we study the problem of flow through a channel with two central electrodes* of length 2λ connected through a load R. The corresponding regions in the z-, w- and t-planes are illustrated in Figs. 2 and 3. The solution to the problem is given by the formulas

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* The problem of the spreading of the current in such a channel due to an applied external potential difference in the absence of a magnetic field is dealt with in [1].

$$w = \frac{\mathscr{E}}{(2 + R\sigma\alpha)\,K(k)} \int_0^t \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}} \qquad (2.4)$$

$$t = i \exp\frac{\pi(z + \lambda)}{2\delta}, \qquad k = \exp -\frac{\lambda\pi}{\delta}, \qquad k' = \sqrt{1 - k^2}, \quad \alpha^{-1} = \frac{K(\kappa)}{K(k')}$$

The potential of the upper electrode ϕ_1 and total current across the load J are

$$\varphi_1 = \frac{\mathscr{E}R\sigma\alpha}{2(2 + R\sigma\alpha)}, \qquad J = \frac{\mathscr{E}\sigma\alpha}{2 + R\sigma\alpha} \qquad (2.5)$$

The current density in the channel and the Joule dissipation $q(x, y)$ per unit volume can be expressed thus:

$$\frac{1}{\sigma} j_x = -f(x, y)\sin\left[\frac{1}{2}(\beta_1 + \beta_2 - \pi\delta^{-1}y)\right], \qquad \frac{1}{\sigma} j_y = f(x, y)\cos\left[\frac{1}{2}(\beta_1 + \beta_2 - \pi\delta^{-1}y)\right]$$

$$\beta_{1,2} = \arg\{1 + \cos(\pi\delta^{-1}y)\exp\pi\delta^{-1}(x \pm \lambda), \quad \sin(\pi\delta^{-1}y)\exp\pi\delta^{-1}(x \pm \lambda)\} \qquad (2.6)$$

$$f(x, y) = \frac{\pi\mathscr{E}}{2\sqrt{2}\delta(2 + R\sigma\alpha)K(k)}\exp\frac{\lambda\pi}{2\delta}\cdot\{[\cosh\pi\delta^{-1}(x + \lambda) + \cos\pi\delta^{-1}y]\times$$

$$\times[\cosh\pi\delta^{-1}(x - \lambda) + \cos\pi\delta^{-1}y]\}^{-0.25}, \qquad q(x, y) = \sigma f^2(x, y)$$

In Formulas (2.4) to (2.6) the quantity $K(k)$ is a complete elliptic integral of the first type. Function $\alpha(\lambda)$ increases monotonically with increase in the argument, whilst $\alpha(0) = 0$, $\alpha(\infty) = \infty$. It is easy to demonstrate the following:

1. The function $j_x(x, y)$ is odd, the function $j_y(x, y)$ is even in its arguments.

2.
$$\begin{aligned}
&j_x = 0, \quad j_y = \sigma f(x, 0) && \text{for } y = 0, \quad 0 \leqslant x < \infty\\
&j_x = 0, \quad j_y = \sigma f(x, \delta) && \text{for } y = \delta, \quad 0 \leqslant x < \lambda\\
&j_x = -\sigma f(x, \delta), \quad j_y = 0 && \text{for } y = \delta, \quad x > \lambda\\
&j_x = 0, \quad j_y = \sigma f(0, y) && \text{for } x = 0, \quad 0 \leqslant y \leqslant \delta
\end{aligned}$$

3. Within the region $x \geqslant 0$, $0 \leqslant y \leqslant \delta$ the currents $j_y \geqslant 0$, $j_x \leqslant 0$. Several streamlines are shown diagrammatically in Fig. 2.

4. At the points $y = \pm\delta$, $x = \pm\lambda$, the function $w(z)$ is no longer analytic. Therefore, for instance, the dissipation $q(x, y)$ at those points is infinite. However, the overall characteristics, namely the current leaving through the electrode section adjacent to one of these points, and the dissipation in the neighborhood of these points when the dimension of the corresponding electrode section and the radius of the region considered vanish, both tend to zero. The total dissipation in the channel

is finite.

It is evident from Formulas (2.5) that the magnitude J of the current increases with increase in the electrode dimension, while the electrical conductivity decreases when the external resistance increases (Fig.4).

In the case where $\lambda \to \infty$, i.e. when the channel walls are electrodes all along their length, the potential on the upper (lower) electrode, and the current, are respectively

$$\varphi(x, \pm\delta) = \pm \frac{1}{2}\mathscr{E}, \qquad J = \mathscr{E}R^{-1}$$

In this case there is no Joule loss in the channel.

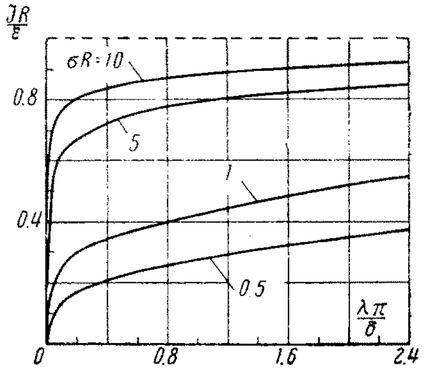

Fig. 4.

When $\lambda \to 0$, i.e. when the walls are insulating all along their length and have point conducting terminals (exits) at $x = 0$, $y = \pm\delta$, the potential at the terminals and the current obtained are zero. Over the rest of the wall surface the potential is

$$\varphi(x, \pm\delta) = \pm \frac{1}{2}\mathscr{E}$$

The current in the channel is zero, and the electric charge is separated.

3. Now suppose that the magnetic field for $a_\nu \leqslant x \leqslant b_\nu$, $\nu = 1, \ldots, n$ may be expressed as an arbitrary function $B = B_\nu(x)$, $B_\nu(a_\nu) = B_\nu(b_\nu) = 0$ and vanishes outside the electrodes. We assume that the velocity of the medium is constant everywhere. The function $\phi(x, y)$, therefore, will be harmonic and it is possible to construct an analytic function $w = \phi + i\psi$. As the potential $(\phi(x \pm \delta) = \pm \phi_\nu)$ is constant over the electrodes, while because of the condition $B \equiv 0$ the imaginary part of the function w is constant on the insulation, the problem can again be solved by conformal representation of the regions on the inside of a polygon in the w-plane with sides parallel to the coordinates, as in Section 2. The change, $\psi_\nu = \psi(a_\nu, \delta) - \psi(b_\nu, \delta)$ in the function $\psi(x, y)$ on the electrodes, is related to the quantity ϕ_ν by condition (1.5)

$$G_\nu - \psi_\nu = 2\varphi_\nu / R_\nu\sigma, \qquad G_\nu = \frac{1}{c} V \int_{a_\nu}^{b_\nu} B_\nu(x)\, dx$$

The potential vanishes at infinity.

Let us now deal with a channel with central electrodes of length 2λ (Fig. 5). The corresponding region in the w-plane is shown in Fig. 6. The solution is given by the formulas

$$w = -\frac{R\sigma G}{(2+R\sigma\alpha)K(k)}\int_0^t \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad G = \frac{1}{c}V\int_{-\lambda}^{+\lambda} B dx \tag{3.1}$$

The potential at the upper electrode ϕ_1 and the resulting current J are

$$\varphi_1 = \frac{R\sigma G}{2+R\sigma\alpha}, \qquad J = \frac{2\sigma G}{2+R\sigma\alpha} \tag{3.2}$$

and the currents j_x and j_y are expressible as

$$\frac{1}{\sigma} j_x = \tau(x,y)\sin\left[\frac{1}{2}(\beta_1+\beta_2-\pi\delta^{-1}y)\right]$$

$$\frac{1}{\sigma} j_y = -\tau(x,y)\cos\left[\frac{1}{2}(\beta_1+\beta_2-\pi\delta^{-1}y)\right] + \frac{1}{c}VB(x). \tag{3.3}$$

$$\tau(x,y) = \frac{\pi R\sigma G}{2\sqrt{2}\,\delta(2+R\sigma\alpha)K(k)}\exp\frac{\lambda\pi}{2\delta}\times$$

$$\times\{[\cosh\pi\delta^{-1}(x+\lambda)+\cos\pi\delta^{-1}y]\,[\cosh\pi\delta^{-1}(x-\lambda)+\cos\pi\delta^{-1}y]\}^{-0.25}$$

In Formulas (3.1) to (3.3) the quantities β_1, β_2, a, k, $K(k)$ and t are expressed in the same way as in Section 2. Assume the function $B(x)$ to be even and $B(x) \leqslant B(0)$; it is easy to check the following:

1) the function $j_x(x, y)$ is odd, whilst $j_y(x, y)$ is even;

2) when $x > \lambda$, $0 \leqslant y \leqslant \delta$, the current $j_y \leqslant 0$ for $x \geqslant 0$, $0 \leqslant y \leqslant \delta$; current $j_x \geqslant 0$;

3) on the axis $x = 0$ the current $j_y > 0$.

It follows that for any value $0 \leqslant y_1 \leqslant \delta$ a point $P(x_1, y_1)$, $x_1 < \lambda$ can be found for which $j_y = 0$ while the streamlines have a horizontal tangent. The appearance of several streamlines is shown on Fig. 5.

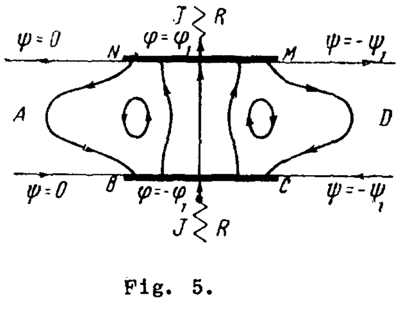

Fig. 5.

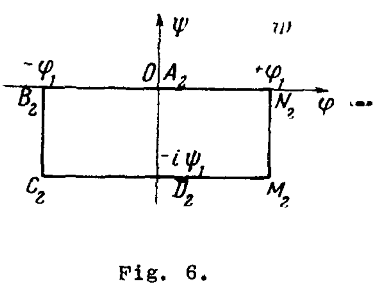

Fig. 6.

It is evident from Formulas (3.2) that the current across the external load increases with increase in the conductivity of the medium and the magnitude of the field, but decreases when the resistance of the external load is increased. When the length of the electrodes is increased, keeping σ, G, R constant, the current output is reduced, and, finally, when $\lambda \to \infty$ no current appears in the external circuit. For high values of λ, from (3.3) we have approximately

$$j_x \equiv 0, \qquad \frac{1}{\sigma} j_y = \frac{1}{c} VB(x) - \frac{1}{2c\lambda} V \int_{-\lambda}^{+\lambda} B\,dx$$

At positions where the magnetic field is relatively strong, currents flow from the lower electrode to the upper one; they then flow along the electrodes and close the circuit through the remaining portion of the channel where the magnetic field is almost zero.

In the other limiting case $\lambda \to 0$ the current and the potential at the terminals are

$$J = G\sigma, \qquad \varphi_1 = \frac{1}{2} R\sigma G$$

On the remaining parts of the walls the potential is zero. In this case the magnetic field $B(x) = G\delta(x)$ (where $\delta(x)$ is a delta function).

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Translated by V.H.B.